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The Fundamental Theorem of Classical Mechanics

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Assuming two given time-independent Newtonian systems of the same dimensions, each of the two systems including its own given set of time-independent generalized Poisson Brackets and a time-independent Hamiltonian, there always locally exists a one-to-one function from variables of one system to the variables of the other system, such that it transforms equations of motion of the first system into equations of motion of the second system, the Poisson Brackets of the first system into the Poisson Brackets of the second system, and the Hamiltonian of the first system into the Hamiltonian of the second system.

One interpretation of the above is that all mechanical systems of the same dimension are locally identical, and the variety of systems we observe in the real world is due only to the fact that we use different systems of variables when making our observations.

I. INTRODUCTION

There seemingly exists a large number of possible mechanical systems of a given dimension that we observe both in physical world as well as among purely mathematical models. This variety leads to questions about the properties of mathematical models used for describing these systems, specifically about the existence of Hamiltonians, Lagrangians etc., for at least some of them.

In this work we want to show that such a large variety of situations is apparent only, not real, and that it is due to the use of different variables. We want to show that it is the use of different variables that creates an impression of the existence of so many different possible systems.

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Specifically, we are going to show that between any two systems of the same dimension there locally exists a one-to-one function that may be described as a change of variables, such that differential equations of motion of the first system are transformed by the function into the equations of motion of the second system (and therefore also all the possible motions in the first system are transformed into all possible motions in the other system), while simultaneously the Poisson Brackets of the first system are transformed into Poisson Brackets of the second system, and the Hamiltonian of the first system is transformed into the Hamiltonian of the second system. Since arguably the equations of motion, the Poisson Brackets, and the Hamiltonian make for the totality of what we call the "physics" of a system, we can claim that locally just one system exists, and the apparent differences between systems are due only to the use of different variables, and therefore are not essential.

The organization of our presentation is as follows:

In section II, we specify the mechanical systems we will consider. We also define the Poisson Brackets and Hamiltonians that are consistent with a given mechanical system.

In section III, we look at the Box variables and their relation to the Poisson Brackets and the Hamiltonian consistent with the equations of motion in Box variables.

In section IV, we show that so called Darbox variables, variables that are both Box variables and canonical variables, exist for any mechanical system. We also show that in the Darbox variables only the first "position" variable is changing in time, with time derivative equal to 1, while all other variables are constant in time. We also show that in these variables the first "momentum" variable is always equal to the original Hamiltonian of the system.

In section V, we formulate and prove the Fundamental Theorem of Classical Mechanics.

In section VI, we briefly describe the most basic mathematical consequences of the Fundamental Theorem of Classical Mechanics.

In section VII, we briefly describe the most basic science-philosophical consequences of the Fundamental Theorem of Classical Mechanics.

In section VIII, we present an example of the Fundamental Theorem of Classical Mechanics, by providing an explicit change of variables between two physically very different physical systems.

In section IX, we make some closing comments.

II. THE NEWTONIAN DYNAMICAL SYSTEM AND ITS ASSOCIATED HAMILTONIAN FORMALISMS

Consider a given system of particles described locally by n generalized spatial coordinates $(x_1,...,x_n)$. Assume that their possible motions satisfy the following Newtonian system of n second-order, time-independent, equations of motion:

$$\ddot{x}_i = R_i(x_j, \dot{x}_j), \quad i, j = 1, \ldots, n. \tag{1}$$

By introducing velocity variables $(v_1, \ldots, v_n)$, defined by $v_i = \dot{x}_i$, the equations of motion (1) can be rewritten as:

$$\begin{aligned} \dot{x}_i &= v_i \quad i = 1, \ldots, n, \\ \dot{v}_i &= R_i(x_j, v_j), \quad i, j = 1, \ldots, n. \end{aligned} \tag{2}$$

At this point, it is convenient to change the notation, just for the remainder of this section. We will introduce $\alpha_i = x_i \;\; i = 1, \ldots, n$, and $\alpha_i = v_{i-n} \;\; i = n+1, \ldots, 2n$. Also, we will use the symbol (α) when referring to these coordinates in aggregate. The equations (2) can then be re-written in the form:

$$\dot{\alpha}_i = E_i(\alpha) \qquad i = 1, \ldots, 2n. \tag{3}$$

We now want to reproduce the Newtonian equations in the Hamiltonian formalism. Let us start with recalling a definition[1)] of dynamically allowed Poisson Brackets for the system (3). A system of Generalized Poisson Brackets $\{\ \ ,\ \ \}$ is called dynamically allowed by the equations (1), (2), and (3), if they satisfy the condition

$$\sum_{k=1}^{2n} \frac{\partial\{\alpha_i, \alpha_j\}}{\partial \alpha_k} \cdot E_k = \{E_i, \alpha_j\} + \{\alpha_i, E_j\} \qquad i, j = 1, \ldots, 2n \tag{4}$$

It can be shown[1)] that condition (4) is equivalent to the existence of a Hamiltonian that will reproduce the equations (2) by the usual Hamilton equations.

$$\begin{aligned} \dot{x}_i &= \{x_i, H\}, \quad i = 1, \ldots, n, \\ \dot{v}_i &= \{v_i, H\}, \quad i = 1, \ldots, n. \end{aligned} \tag{5}$$

where the Poisson Brackets described before are used.

In other words, the Hamiltonian and the Poisson Brackets satisfy the equations

$$\begin{aligned} &\{x_i, H\} = v_i \quad i = 1,\dots,n, \\ &\{v_i, H\} = R_i(x_j, v_j), \quad i, j = 1,\dots,n. \end{aligned} \tag{6}$$

where $R_i(x_j, v_j), \quad i, j = 1,\dots,n$ are taken from equations (2).

The equations (3) will be reproduced by the Hamiltonian equations (6), as

$$\dot{\alpha}_i = \{\alpha_i, H\} \qquad i = 1,\dots,2n. \tag{7}$$

As a result of the equations (5), (6) and (7), for any dynamical variable $f = f(x_i, v_i) \quad i = 1,\dots,n$, we now have

$$\dot{f} = \{f, H\}. \tag{8}$$

The equivalence between the condition (4) and the existence of a Hamiltonian is of "if and only if" type, meaning that if the Hamiltonian giving equations (6) exists, then the Poisson Brackets used in (6) will satisfy the condition (4). Vice versa, if given Poisson Brackets satisfy the condition (4), then a Hamiltonian giving (6) will exists[1)]. As one possible consequence, the regular Poisson brackets, the brackets obtained from any traditional Lagrangian, will be among the dynamically allowed generalized Poisson Brackets, and the usual Hamiltonian obtained from that Lagrangian will be among Hamiltonians we describe here.

III. THE BOX VARIABLES AND THE HAMILTONIAN FORMALISMS

Let us start with a given system of equations of motion (1), (2), and (3), and a given system of Poisson Brackets dynamically allowed by these equations, denoted by $\{\ ,\ \}$. Since the brackets $\{\ ,\ \}$ are dynamically allowed, then a Hamiltonian H also exists.

The Box Theorem[2)] tells us that locally there exists a system of variables $(z_i)\quad i=1,...,2n$

$$z_i = z_i(x_j, v_j),\quad i=1,...,2n,\quad j=1,...,n, \tag{9}$$

such that the equations of motion (2) (or (3)), when expressed by the variables (9), become

$$\begin{aligned} \dot{z}_1 &= 1 \\ \dot{z}_i &= 0,\quad i=2,...,2n. \end{aligned} \tag{10}$$

Expressing the Hamiltonian and the Poisson Brackets in variables (9), we get

$$\begin{aligned} \dot{z}_1 &= \{z_1, H\} \\ \dot{z}_i &= \{z_i, H\},\quad i=2,...,2n. \end{aligned} \tag{11}$$

Comparing (10) and (11), we get

$$\{z_1, H\} = 1, \tag{12}$$

$$\{z_i, H\} = 0,\quad i=2,...,2n. \tag{13}$$

In general, as in any variables, the generalized Poisson Brackets is given as[1)]

$$\{f, g\} = \sum_{i,j=1}^{2n} \frac{\partial f}{\partial z_i} \cdot P_{ij} \cdot \frac{\partial g}{\partial z_j}, \tag{14}$$

where $P_{ij} = P_{ij}(z_k) = \{z_i, z_j\}$ is an antisymmetric matrix that also satisfies other conditions[4)] required by the definition of generalized Poisson Brackets.

We will now show that in the box variables (9) the entries of the matrix P_{ij} are independent of z_1.

The Jacobi identity tells us that for any $i, j = 1, ..., 2n$ we have:

$$\{\{z_i, z_j\}, H\} + \{\{z_j, H\}, z_i\} + \{\{H, z_i\}, z_j\} = 0. \tag{15}$$

Then we have (the Einstein's notation is used below, and δ_{kn}, $k, n = 1, ..., 2n$ is the Kronecker delta):

$$0 = \{\{z_i, z_j\}, H\} + \{1\, or\ 0, z_i\} + \{-1\, or\ 0, z_j\} =$$

$$= \{\{z_i, z_j\}, H\} + 0 + 0 =$$

$$= \{\{z_i, z_j\}, H\} =$$

$$= \frac{\partial\{z_i, z_j\}}{\partial z_k} \cdot P_{km} \cdot \frac{\partial H}{\partial z_m} =$$

$$= \frac{\partial\{z_i, z_j\}}{\partial z_k} \cdot \delta_{kn} \cdot P_{nm} \cdot \frac{\partial H}{\partial z_m} =$$

$$= \frac{\partial\{z_i, z_j\}}{\partial z_k} \cdot \frac{\partial z_k}{\partial z_n} \cdot P_{nm} \cdot \frac{\partial H}{\partial z_m} =$$

$$= \frac{\partial\{z_i, z_j\}}{\partial z_k} \cdot \{z_k, H\} =$$

$$= \frac{\partial\{z_i, z_j\}}{\partial z_k} \cdot \delta_{k1} =$$

$$= \frac{\partial\{z_i, z_j\}}{\partial z_1}.$$

So we have

$$\frac{\partial\{z_i, z_j\}}{\partial z_1} = 0, \quad i, j = 1,..., 2n. \tag{16}$$

We may also notice that because of (16) all the entries of the Poisson Brackets matrix

$P_{ij} = \{z_i, z_j\}$ are time independent. This is because

$$\dot{P}_{ij} = \frac{\partial\{z_i, z_j\}}{\partial z_k} \cdot \dot{z}_k = \frac{\partial\{z_i, z_j\}}{\partial z_k} \cdot \delta_{k1} = \frac{\partial\{z_i, z_j\}}{\partial z_1} = 0.$$

IV. THE DARBOX VARIABLES

Let us now start with box variables, generalized Poisson Brackets, and a Hamiltonian described in the previous section. We will now introduce some changes of variables to derive the most convenient, in our opinion, variables to describe that system.

First, please notice that the Hamiltonian must be explicitly dependent on at least one of the variables $z_i, \quad i = 2,..., 2n,$ because if H would only be dependent on z_1, we would have

$$\{z_1, H\} = \sum_{i,j=1}^{2n} \frac{\partial z_1}{\partial z_i} \cdot P_{ij} \cdot \frac{\partial H}{\partial z_j} = \sum_{i,j=1}^{2n} \delta_{1i} \cdot P_{ij} \cdot \frac{\partial H}{\partial z_j} = \sum_{i,j=1}^{2n} \delta_{1i} \cdot P_{ij} \cdot \delta_{j1} \cdot \frac{\partial H}{\partial z_j} = P_{11} \cdot \frac{\partial H}{\partial z_1} = 0,$$

and this would be contradictory to $\{z_1, H\} = 1.$

Assume, without the loss of generality, that H is explicitly dependent on z_2. Then we can replace z_2 by H in the set of box variables. It means we introduce the new set of variables

$Y_i, \quad i = 1,..., 2n,$ defined as:

$Y_1 = z_1,$

$$Y_2 = H, \tag{17}$$

$$Y_j = z_j, \quad j = 3,...,2n.$$

In these variables, using (12), (13) and (17) we have:

$$\begin{aligned} &\{Y_1, Y_2\} = 1, \\ &\{Y_j, Y_2\} = 0, \quad j = 2,...,2n. \end{aligned} \tag{18}$$

The new variables are still box variables, since we have

$$\begin{aligned} &\dot{Y}_1 = \dot{z}_1 = \{z_1, H\} = 1, \\ &\dot{Y}_2 = \dot{H} = \{H, H\} = 0, \\ &\dot{Y}_j = \dot{z}_j = \{z_j, H\} = 0, \quad j = 3,...,2n. \end{aligned} \tag{19}$$

By the same argument as in the previous section, the variables $Y_i, \quad i = 1,...,2n,$ satisfy an equations analogous to equation (16), here written as:

$$\frac{\partial\{Y_i, Y_j\}}{\partial Y_1} = 0, \quad i, j = 1,...,2n. \tag{20}$$

As before, (20) means that the entries $P_{ij}, \quad i, j = 1,...,2n,$ of the Poisson Brackets matrix do not contain the variable Y_1.

Let us now introduce a new real parameter τ and introduce a system of ordinary differential equations on the variables $Y_i, \quad i = 2,...,2n,$ defined by:

$$\frac{dY_i}{d\tau} = \{Y_i, Y_1\}, \quad i = 2,...,2n. \tag{21}$$

Because of (20), the right sides of (21) do not contain the variable Y_1. By our construction, the left side of (21) does not contain the variable Y_1 either. Therefore, we have a set of $2n-1$ self-contained equations for $2n-1$variables in the form:

$$\frac{dY_i}{d\tau} = f_i(Y_2, Y_3, ..., Y_{2n}), \quad i = 2, ..., 2n.$$

Then, using the Box Theorem again, there exists an invertible change of variables

$$Z_i = Z_i(Y_2, Y_3, ..., Y_{2n}), \quad i = 2, ..., 2n,$$

such that

$$\begin{aligned} &\frac{dZ_2}{d\tau} = 1, \\ &\frac{dZ_i}{d\tau} = 0, \quad i = 3, ..., 2n. \end{aligned} \tag{22}$$

Therefore, from (21) we have

$$\begin{aligned} &\{Z_2, Y_1\} = 1, \\ &\{Z_i, Y_1\} = 0, \quad i = 3, ..., 2n. \end{aligned} \tag{22}$$

We also have

$$\{Z_i, Y_2\} = 0, \quad i = 2, ..., 2n. \tag{23}$$

This is because each $Z_i, \quad i = 2, ..., 2n,$ is a function of $(Y_2, Y_3, ..., Y_{2n})$ and each (18) tells us that

$$\{Y_j, Y_2\} = 0, \quad j = 2, ..., 2n. \tag{24}$$

Then any function, using the earlier variables, gives:

$$\{G, Y_2\} =$$

$$= \sum_{i,j=1}^{2n} \frac{\partial G}{\partial z_i} \cdot P_{ij} \cdot \frac{\partial Y_2}{\partial z_j} =$$

$$= \sum_{k=2}^{2n} \sum_{i,j=1}^{2n} \frac{\partial G}{\partial Y_k} \cdot \frac{\partial Y_k}{\partial z_i} \cdot P_{ij} \cdot \frac{\partial Y_2}{\partial z_j} =$$

$$= \sum_{k=2}^{2n} \frac{\partial G}{\partial Y_k} \cdot \sum_{i,j=1}^{2n} \frac{\partial Y_k}{\partial z_i} \cdot P_{ij} \cdot \frac{\partial Y_2}{\partial z_j} =$$

$$= \sum_{k=2}^{2n} \frac{\partial G}{\partial Y_k} \cdot \{Y_k, Y_2\} =$$

$$= \sum_{k=2}^{2n} \frac{\partial G}{\partial Y_k} \cdot 0 =$$

$$= 0$$

Now we complete the variables by defining

$$Z_1 = Y_1.$$

So now we have new variables that satisfy:

$$\begin{aligned} &\{Z_1, Z_2\} = 1, \\ &\{Z_1, Z_i\} = 0, \quad i = 3, \ldots, 2n. \end{aligned} \tag{25}$$

Also:

$$\begin{aligned} &\{Z_1, H\} = 1, \\ &\{Z_i, H\} = 0, \quad i = 2, \ldots, 2n. \end{aligned} \tag{26}$$

Finally, in variables $Z_i,\ i = 1, \ldots, 2n,$ the Hamiltonian H must explicitly depend on Z_2. If it was not, then we would have, using $z_i,\ i = 1, \ldots, 2n,$ as variables:

$$\{Z_1, H\} =$$

$$= \sum_{i,j=1}^{2n} \frac{\partial Z_1}{\partial z_i} \cdot P_{ij} \cdot \frac{\partial H}{\partial z_j} =$$

$$= \sum_{k=1}^{2n} \sum_{i,j=1}^{2n} \frac{\partial Z_1}{\partial z_i} \cdot P_{ij} \cdot \frac{\partial Z_k}{\partial z_j} \cdot \frac{\partial H}{\partial Z_k} =$$

$$= \sum_{k=1}^{2n} \{Z_1, Z_k\} \cdot \frac{\partial H}{\partial Z_k} =$$

$$= \sum_{k=1}^{2n} \delta_{2k} \cdot \frac{\partial H}{\partial Z_k} =$$

$$= \frac{\partial H}{\partial Z_2} = 0,$$

which is in contradiction with (25). So $\frac{\partial H}{\partial Z_2} \neq 0$. Therefore, H can replace Z_2 among our variables. So now we have box variables made of $(Z_1, H, Z_3, Z_4, ..., Z_{2n})$. The Poisson Brackets on this system are:

$$\begin{aligned} &\{Z_1, H\} = 1, \\ &\{Z_1, Z_i\} = 0, \quad i = 3, ..., 2n, \\ &\{Z_i, H\} = 0, \quad i = 3, ..., 2n. \end{aligned} \tag{27}$$

Now, let us look at variables $(Z_3, Z_4, ..., Z_{2n})$. They make, by themselves, a system with generalized Poisson Brackets (and id does not matter if they allow a Hamiltonian in a previously defined sense). Then, using the Darboux Theorem[3)], we can replace them by new variables

$$\begin{aligned} &Q_i = Q_i(Z_j), \\ &P_i = P_i(Z_j), \quad i = 2, ..., n, \quad j = 3, ..., 2n. \end{aligned} \tag{28}$$

such that

$$\begin{aligned} &\{Q_i, Q_j\} = 0, \\ &\{P_i, P_j\} = 0, \\ &\{Q_i, P_j\} = \delta_{ij}, \qquad i, j = 3, ..., n. \end{aligned} \tag{29}$$

Then, introducing

$$\begin{aligned} &Q_1 = Z_1, \\ &P_1 = H, \end{aligned} \tag{30}$$

Concluding all work in the first three sections, we have proven the existence of local invertible variables change

$$Q_i = Q_i(x_j, v_j),$$
$$P_i = P_i(x_j, v_j), \quad i, j = 1, \ldots, n, \tag{31}$$

such that the new variables are canonical with respect to the original Poisson Brackets, meaning

$$\{Q_i, Q_j\} = 0,$$
$$\{P_i, P_j\} = 0, \tag{32}$$
$$\{Q_i, P_j\} = \delta_{ij}, \qquad i, j = 1, \ldots, n,$$

and in which the first momenta variable is equal to the original Hamiltonian, meaning

$$H = P_1. \tag{33}$$

We will take a freedom of naming these variables. To honor Jean Gaston Darboux of the theorem used above, and to signify the importance of the Box Theorem in arriving at them, we propose to name these variables Dar-Box variables, or simply Darbox Variables.

V. THE FUNDAMENTAL THEOREM OF CLASSICAL MECHANICS

The fundamental theorem of classical mechanics may be formulated as follows:

For any two mechanical systems of the same total dimension, each of the two systems consisting of differential equations of motion, a Poisson Brackets system consistent with these equations of motion, a Hamiltonian producing these equations of motion via the Poisson Brackets, and expressed in its own position-velocity variables, there exists a locally defined one-to-one function relating the variables of the two systems that transforms the differential equations of motion of one system into the equations of motion of the other system, the Poisson Brackets of one system into the Poisson Brackets of the other system, and the Hamiltonian of one system into the Hamiltonian of the other system.

The proof of the Fundamental Theorem of Classical Mechanics is simple when using the results from the earlier sections. To create the function described in the Fundamental Theorem of Classical Mechanics we start with position-velocity variables of one system, then change to the Darbox coordinates of that system, then go to the Darbox coordinates of the other system by relating the respective Darbox coordinates of both systems one by one, and then go from the Darbox coordinates of the second system to the position-velocities variables of the second system. This defines the function relating two systems.

Since the Darbox variables are box variables, the differential equations of motion of one system change in an obvious way into the equations of motion of the second system. Since the Darbox variables are canonical variables, the Poisson Brackets of one system obviously transform to the Poisson Brackets of the other system. And since the Hamiltonian of each system is equal to the first momentum variable in the Darbox variables, the Hamiltonian of one system is transformed to the Hamiltonian of the other system.

VI. BASIC MATHEMATICAL CONSEQENCES OF THE FUNDAMENTAL THEOREM OF CLASSICAL MECHANICS

The Fundamental Theorem allows a new approach to mechanical systems. Until now we had many mechanical systems of the same dimension, each with its own underlying manifold representing the configuration space. Now the approach may be different: for each dimension we have only one underlying “Hamifold” equipped with generic Poisson-Hamiltonian structure, and then we impose a specific position-velocity local chart on it to observe a specific mechanical system.

With this approach, a one-dimensional harmonic oscillator and a one-dimensional free-falling particle, both equipped with customary Poisson-Hamiltonian structure, are actually the same system. They are just observed by two different observers, each using a different variable system.

Changing our thinking from a separate manifold for each physically distinct system into locally just one "Hamifold" that includes all systems of the same dimensions would be a quite significant change. The differences between possible global "Hamifolds" of the same dimensions could possibly be restricted to global topological differences.

Some other, less profound results, are:

1) Starting with a system of differential equations of motion in even dimensions, with no given Poisson Brackets and a Hamiltonian, we can characterize all possible Poisson Brackets systems with all possible respective Hamiltonians by listing all possible Box variables for these equations, then declare each Box variables system to be Darbox variables system by making the first half of the Box variables to be "positions" of the Darbox variables, and the second half of the Box variables to be the "momenta" of the Darbox variables, and then imposing the Poisson Brackets using the Darbox variables, and imposing the Hamiltonian to be the first "momentum" of the system. This way we will cover all possible Hamilton-Poisson systems for the given differential equations of motion, although different Box variables may give the same Hamilton-Poisson system, so it is not a one-to-one correspondence. Since two different original equations systems of

the same dimension still have identical possible Box variables systems, then also all allowed Hamilton-Poisson systems will be identical for both.

2) We can impose an existing Hamilton-Poisson system from one differential equation system onto another, by associating Darbox variables of the first system with a Box variable system of the second.
3) We can possibly classify all Lagrangians for given dimension of a mechanical system. The question, does a Lagrangian exists for a given system of differential equations of motion, makes no sense anymore, because no matter what the given differential equations are, these equations always represent the same system, just observed using different variables. So, the question should be replaced by the question, in which variables the Lagrangian exist? Obviously, the Darbox variables would not allow a Lagrangian, since the "time" derivatives of all variables are constant there, and the Lagrangian needs to use non-zero derivatives as its building blocks. But we can always change to the "free particle" variables, and then we have the regular Lagrangian which, in a sense, is a Lagrangian for that mechanical system in these specific variables. Obviously, we can change to some other variables that will have other Lagrangians. So, our system has multiple Lagrangians, each in a different set of variables.

VII. SOME SCIENCE-PHILOSOPHICAL CONSEQENCES OF THE FUNDAMENTAL THEOREM OF CLASSICAL MECHANICS

The great success of Newtonian Mechanics in explaining both phenomena experienced on Earth, as well as observed celestial motions, established the commonly accepted approach to scientific view of the Universe. While today the Newtonian Mechanics is no longer considered to be the sole basis of valid and complete theory of the Universe, we generally still retain its big picture, according to which the Universe is a place governed by some basic rules, called the Laws of Physics, and the events we observe represent one particular solution allowed by these Laws , originating from some set of initial (or just earlier) conditions.

The Fundamental Theorem of Classical Mechanics changes this picture. If we assume that a Hamilton-Poisson system of a very high, but still finite dimension, is a proper macroscopic model of the Universe, then the Fundamental Theorem tells us that all motions are possible in that Universe, and not only are they possible, but also they actually happen. We do not see them, since we observe the Universe using just some among all possible variable systems, and in these variable systems only limited set of events presents itself to us.

In the current approach, the origin of the Laws of Physics is not clear, as it is not the source of specific initial conditions of a state of the known Universe. The Fundamental Theorem tell us that the Laws of Physics are not universal rules. They are a mere consequence of the specific variables we use to describe the Universe. They require no explanation of their origin, since in different variables they would be different. In other words, anything is possible, and that anything is actually happening, just in variables that we are not using. Similarly, initial conditions of the Universe that we observe require no explanation, since they would appear to be completely different in different variables. Any possible initial conditions are reality in properly chosen variables.

Notice that this model could possibly replace or give additional insight into the Multiverse model postulated by some physicists.

Finally, let us include a comment about the role of variables in our understanding of the Classical Mechanics. Specific variables were very important for any description of what was happening around us from the beginnings of the scientific method and were extensively used by Galileo, Newton and others who followed. Later, the introduction of generalized variables, the Lagrangian-Hamiltonian approach, and the concept of a manifold pushed specific variables away from the forefront of Classical Mechanics. The models were supposed to be independent of variables in which they were presented. Thus, are variables important or not from the point of view of the Fundamental Theorem? It seems that the answer is, "both." The variables are not important in the sense that any variables can describe any system, and none are better than others. But they are also extremely important, because they decide what specific Laws of Physics we have when describing the Universe. Laws of Physics are tied to the choice of specific variables and are different in different variables.

VIII. AN EXAMPLE

Let us start with a harmonic oscillator with regular Hamiltonian and regular Poisson Brackets. We have

$$\dot{x} = v, \qquad \dot{v} = -x, \tag{34}$$

as the differential equations of motion and

$$H = \frac{v^2}{2} + \frac{x^2}{2}, \tag{35}$$

$$\{x, v\} = 1, \tag{36}$$

as the Hamiltonian and the Poisson Brackets. It is easy to see that

the equations:

$$\begin{aligned} \dot{x} &= \{x, H\}, \\ \dot{v} &= \{v, H\}, \end{aligned} \tag{37}$$

reproduce the equations of motion (34) as expected.

Now consider new variables, (y, w), defined as:

$$\begin{aligned} y &= \frac{1}{2}\left[\arctan\left(\frac{x}{v}\right)\right]^2 - \frac{v^2}{2} - \frac{x^2}{2}, \\ w &= \arctan\left(\frac{x}{v}\right). \end{aligned} \tag{38}$$

The formulas (38) can be locally inverted, giving:

$$\begin{aligned} x &= \sqrt{w^2 - 2y} \cdot \sin w, \\ v &= \sqrt{w^2 - 2y} \cdot \cos w. \end{aligned} \tag{39}$$

Let us now express the equations of motion in new variables. The chain rule tells us that:

$$\begin{aligned} \dot{y} &= \frac{\partial y}{\partial x} \cdot \dot{x} + \frac{\partial y}{\partial v} \cdot \dot{v}, \\ \dot{w} &= \frac{\partial v}{\partial x} \cdot \dot{x} + \frac{\partial v}{\partial v} \cdot \dot{v}. \end{aligned} \tag{40}$$

Calculating partial derivatives from (38), using time derivatives from (34) and then replacing variables (x, y) with variables (y, w), after somewhat tedious simplification gives us:

$$\begin{aligned} \dot{y} &= w, \\ \dot{w} &= 1. \end{aligned} \tag{41}$$

Then, since we have the standard Poisson brackets $\{x, v\} = 1$, calculating the Poisson Brackets of (y, w) can be done as:

$$\{y, w\} = \frac{\partial y}{\partial x} \cdot \frac{\partial w}{\partial v} - \frac{\partial y}{\partial v} \cdot \frac{\partial w}{\partial x}. \tag{42}$$

Calculating the partial derivatives using (38), substituting into (42), after some simplification gives:

$$\{y, w\} = 1. \tag{43}$$

Finally, starting with the Hamiltonian (35), substituting (39) and simplifying, gives:

$$H = \frac{w^2}{2} - y. \tag{44}$$

The equations of motion (41), the Poisson Brackets (43), and the Hamiltonian (44) represent a free falling (upward) body in one spatial dimension.

So as predicted by the Fundamental Theorem of Classical Mechanics, two quite different physically systems, a free-falling body and a harmonic oscillator, are nevertheless locally identical from the Hamilton-Poisson point of view.

IX. FINAL REMARKS

In conclusion, we would like to state that the most important fact described in this work is that in a Poisson-Hamiltonian model anything happens, any time evolution may happen, and it is in fact happening in a way parallel to infinitely many other time evolutions of the very same system. The evolution that we actually observe and the possible laws that govern that evolution are solely dependent on the variables chosen for observations, not on the nature of the system itself.

We would also like to stress that this work does not represent a new theory or a new model. We took the formalism created by Lagrange, Hamilton, and Poisson more than 200 years ago and just followed that formalism to some new (at least to us), conclusions. These facts were present in their formalism from its beginnings more than 200 years ago; we just were not aware of them.

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